

Root systems from Toric Calabi-Yau Geometry. Towards new algebraic structures and symmetries in physics?

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ABSTRACT: The algebraic approach to the construction of the reflexive polyhedra that yield Calabi-Yau spaces in three or more complex dimensions with K3 fibres reveals graphs that include and generalize the Dynkin diagrams associated with gauge symmetries. In this work we continue to study the structure of graphs obtained from CY_3 reflexive polyhedra. We show how some particularly defined integral matrices can be assigned to these diagrams. This family of matrices and its associated graphs may be obtained by relaxing the restrictions on the individual entries of the generalized Cartan matrices associated with the Dynkin diagrams that characterize Cartan-Lie and affine Kac-Moody algebras. These graphs keep however the affine structure, as it was in Kac-Moody Dynkin diagrams. We presented a possible root structure for some simple cases. We conjecture that these generalized graphs and associated link matrices may characterize generalizations of these algebras.

KEYWORDS: .

1. Introduction

The final objective of this work is no short of ambition: to discuss the question of what is nature of the ultimate, or at least of the next, set of symmetries of the Standard Model. The success of the use of Cartan-Lie Algebras and their direct products in the GSW model got full confirmation by the discovery of the intermediating bosons via the observation of neutral currents and is now beyond any doubt. However the efforts dedicated to the extension of the SM group of symmetries, always in the frame of Cartan-Lie algebras and with the objective of, for example, reducing the number of free parameters appearing in the theory, has not lead to the same level of success. Equally, the attempts of Grand Unified theories to englobe the direct product of the symmetries of the SM in some larger group has become problematic, with not clear new predictions and with a systematic too large proton instability.

At a very basic level, and without any obvious direct interest to SM, Cartan-Lie symmetries are closely connected to the geometry with symmetric homogeneous spaces whose classification was performed by Cartan himself in the classic epoch. Furtherly, the developing of an alternative geometry of non-symmetric spaces appeared later, their classification was suggested in 1955 by Berger using holonomy theory [1]: there are some infinite series of spaces with holonomy groups $SO(n)$, $U(n)$, $SU(n)$, $Sp(n) \times Sp(1)$, $Sp(n)$ and, in addition, some exceptional spaces with holonomy groups $G(2)$, $Spin(7)$, $Spin(16)$.

The Standard Model does not provide, at least visibly to us, any clue on how to attack the problem of the nature of symmetries at this very basic geometry level. This is not the case for the superstring theories and general Kaluza-Klein scenarios, for example the compactification of the heterotic string leads to the classification of states in a representation of the Kac-Moody algebra of the gauge group $E_8 \times E_8$ or $Spin(32)/Z_2$. In another words, heterotic superstring discovered for physics the 6-dimensional Calabi-Yau space, an example of non-symmetrical space having the $SU(3)$ group of holonomy [2]. It has also been shown [3] that group and algebra theory appear at the root of generic two-dimensional conformal field theories (CFT). The basic ingredient here is the central extension of infinite dimensional Kac-Moody algebras. There is a clear connection between these algebraic and geometric generalizations. Affine Kac-Moody algebras are realized as the central extensions of Loop algebras: the set of mappings on a compact manifold, for example S^1 , which take value on a finite-dimensional Lie algebra. In summary, Superstring theory intrinsically contains a number of other infinite-dimensional algebraic symmetries such as the Virasoro algebra associated with conformal invariance and to generalizations of Kac-Moody algebras themselves, such as hyperbolic and Borcherd algebras.

Remaining with Calabi-Yau spaces, Dynkin diagrams (or Coxeter-Dynkin diagrams) which are in one-to-one correspondence not only to Cartan-Lie but also Kac-Moody algebras has also been observed through the technique of resolution of singularities. The rich singularity structure of some examples of non-symmetrical Calabi-Yau spaces gives us here another opportunity to re-obtain infinite dimensional, affine, Kac-Moody symmetries. The Cartan matrix of affine Kac-Moody groups is identified with the intersection matrix of the union of the complex projective lines resulting from the blow-up of the singularities. This

is indeed the case of $K3 \equiv CY_2$ where the classification of degeneration of their fibers and their associated singularities, leads us to link CY_2 spaces with the infinite and exceptional series $A_r^{(1)}$, $D_r^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ of affine Kac-Moody algebras.

The study of Calabi-Yau spaces appearing in superstring, F and M theories can be approached from the theory of Toric geometry and the so called Batyrev's construction [4] where the concept of reflexive polyhedra appears. This concept of reflexivity or mirror symmetry has been linked [5] to the problem of the duality between superstring theories compactified in different (K3 and CY_3) Calabi-Yau spaces. The same Batyrev construction has also been used to show how subsets of points in these reflexive polyhedra can be identified with the Dynkin diagrams [5–8] of the affine version of the gauge groups appearing in superstring and F-Theory. More explicitly, the gauge content of the compactified theory can be read off from the *dual* reflexive polyhedron of the Calabi-Yau space which is used for the compactification.

For a K3 Calabi-Yau space, subdivision of the reflexive polyhedron in different subsets separated by polygons which are themselves reflexive is equivalent to establish a fibration structure for the space. The fiber simply being the corresponding one to the this intersecting mirror polygon. Any of these parts of the full polyhedron has been called by Candelas and Font a “top” [5]. Thus, reflexive polyhedra when intersected with a plane gives raise to individual reflexive polygons. This intersection allows us to define two parts of the polyhedron: “top” and “bottom” in the nomenclature of Ref.[5] or left and right in what follows in this work.

Calabi-Yau spaces being characterized geometrically by reflexive Newton polyhedra can be enumerated systematically. However, one might going deeper than simple enumeration, it has been recently realized that different reflexive polyhedra are related algebraically via what it has been termed Universal Calabi-Yau Algebra (UCYA). The term “Universal” comes from the fact that includes, beyond binary, ternary and higher-order operations.

The UCYA is particularly well suited for exploring fibrations of Calabi-Yau spaces, which are visible as lower-dimensional slices through higher-dimensional mirror polyhedra or, alternatively, as projections on the reflexive polyhedron. The UCYA approach naturally gives these slice or projection structures. See Fig. 1 in Ref.[9] and its explanation for a description of the elliptic fibration of a K3 space. The Left and Right parts of the reflexive polyhedron corresponds correspond each to a, so called, extended vector. According to the UCYA scheme, the sum, a binary operation, of these two “extended vectors” gives true reflexive vector, describing the full $CY_2 = K3$ manifold. This means that a direct algebraic relation between $K3 (= CY_2)$ and CY_3 spaces is established. This property is completely general: it has been shown previously how the UCYA, with its rich structure of beyond binary operations, can be used to generate and interrelate CY_n spaces of any order. The UCYA provides in addition a complete and systematic description of analogous decompositions or nesting of fibrations in Calabi-Yau spaces of any dimension.

One of the remarkable features of Fig. 1 in Ref.[9] is that each of the right and left set of nodes constitute graphs corresponding to affine Dynkin diagrams: the $E_6^{(1)}$ and $E_8^{(1)}$ diagrams. This is not a mere coincidence or an isolated example. As discussed there, all the elliptic fibrations of K3 spaces found using the UCYA construction feature this

decomposition into a pair of graphs that can be interpreted as Dynkin diagrams.

The purpose of this paper is to deal with the important problem on how to generalize previous results, applicable to K3 only, to any Calabi-Yau space with any dimension and any fiber structure. In first place, we would like to know which Dynkin-like diagrams can be found “digging” in higher dimension Calabi-Yau spaces. As it has firstly been shown in Ref.[9] a bounty of new diagrams, “Berger Graphs”, can be brought to light by this procedure. Our hypothesis is that these graphs correspond, in some undefined way, to some new algebraic structure, as Dynkin diagrams are in one-to-one correspondence with root systems and Cartan matrices in semi-simple Lie Algebras and affine Kac-Moody algebras.

Based on this, our next objective is to construct a similar theory to Kac-Moody algebras where some newly defined generalized Cartan matrix fulfilling extended conditions are introduced. There are plenty of possible, trivial and not so trivial, generalizations of Cartan matrices where the rules for the diagonal and non diagonal entries are modified, it is impossible to find all of them and classify. On the other hand, probably not all them give meaningful, consistent generalizations of Kac-Moody algebras and probably even less of them have interesting implications for physics. One has to find natural, and hopefully physically inspired, conditions on these matrices. The relation of Berger Graphs to Calabi-Yau spaces could be this physical inspiring link.

Having the equivalent of Cartan Matrix we can use standard algebraic tools, definition of a inner product, construction of root systems and their group of transformation etc, which could be helpful in clarifying the meaning and significance of this construction.

The structure of this paper is as follows. In section 2 we show how to extract graphs directly from the polyhedra associated to Calabi-Yau spaces and how, from these graphs define new ones by adding or removing nodes. We include a first step towards the systematic enumeration of rules to build generalized Cartan matrices from the graphs, from this we give an algebraic definition of a new set of matrices generalizing the Cartan-Kac Moody matrices and, finally, rules for the reverse process of obtaining graphs from matrices. In section 3 we present some elementary but illustrative examples of these procedures. Finally we draw some conclusions and conjectures.

2. UCYA and generalized Dynkin diagrams.

One of the main results in the Universal Calabi-Yau Algebra (UCYA) is that the reflexive weight vectors (RWVs) \vec{k}_n of dimension n , which are the fundament for the construction of CY spaces, can obtained directly from lower-dimensional RWVs $\vec{k}_1, \dots, \vec{k}_{n-r+1}$ by algebraic constructions of arity r [10–13]. The dimension of the corresponding vector is $d + 2$ for a Calabi-Yau CY_d space.

For example, the sum of vectors, a binary composition rule of the UCYA, gives complete information about the $(d - 1)$ -dimensional slice structure of CY_d spaces. In the K3 case, the Weierstrass fibered 91 reflexive weight vectors of the total of 95 \vec{k}_4 can be obtained by such binary, or arity-2, constructions out of just five RWVs of dimensions 1,2 and 3.

In an iterative process, we can combine by the same 2-ary operation the five vectors of dimension 1, 2, 3 with these other 95 vectors to obtain a set of 4242 chains of five-dimensional RWVs \vec{k}_5 CY₃ chains. This process is summarized in Fig. 3 in Ref.[9]. By construction, the corresponding mirror CY₃ spaces are seen to possess K3 fibre bundles. In this case, reflexive 4-dimensional polyhedra are also separated into three parts: a reflexive 3-dimensional intersection polyhedron and ‘left’ and ‘right’ skeleton graphs. The complete description of a Calabi-Yau space with all its non-trivial d_i fibre structures needs a full range of n-ary operations where $n_{max} = d + 2$.

It has been shown in the toric-geometry approach how the Dynkin diagrams of affine Cartan-Lie algebras appear in reflexive K3 polyhedra [4–8]. Moreover, along the same lines using examples of the lattice structure of reflexive polyhedra for CY_{*n*} : $n \geq 2$ with elliptic fibres, it has also been shown [10, 12, 13], that there is a correspondence between the five basic RWVs (basic constituents of composite RWVs describing K3 spaces, see section 2 in [9]) and affine Dynkin diagrams for the five ADE types of Lie algebras (A, D series and exceptional E_{6,7,8}).

In each case, a pair of extended RWVs have an intersection which is a reflexive plane polyhedron; each vector from the pair gives the left or right part of the three-dimensional RVW. The construction generalizes to any dimension. In Ref.[9] it was remarked that in the corresponding “left” and right “graphs” of CY_{3,4,..} Newton reflexive polyhedra one can find new graphs with some regularity in its structure.

In principle one should be able to build, classify and understand these regularities of the graphs according to the n-arity operation which originated the construction. For the case of binary or arity-2 constructions: two graphs are possible. In general for any reflexive polyhedron, for a given arity-*r* intersection, it corresponds exactly *r* graphs.

In the binary case, the 2-ary intersection (a plane) in the Newton polyhedra, which correspond to the *eldest* reflexive vector of the series, separate left and right graphs. A concrete rule for the extraction of individual graph points from all possible nodes in the graphs is that they are selected if they exactly belong “on the edges” lying on one side or another with respect to the intersection. In the ternary case, the 3-ary intersection hypersurface is a volume, which separate three domains in the newton polyhedra and three graphs are possible. Individual points are assigned to each graph looking at their position with respect to the volume intersection (see Tab.1 in [9] for some aclaratory examples).

These are graphs directly obtained from the reflexive polyhedron construction. On a later stage, we will define graphs independently of this construction. These graphs will be derived, or by direct manipulation of them, or from generalized Cartan matrices in a purely algebraic fashion. They will basically consist on the primitive graphs extracted from reflexive polyhedra to whom internal nodes in the edges will have been added or eliminated. The nature of the relation, if any, of the graphs thus generated to the geometry of Toric varieties and the description of Calabi-Yau as hypersurfaces on them is related to the possibility of defining viable “fan” lattices. This is an open question, clearly related to the properties of the generalized Cartan matrices, interpreted as a matrix of divisor intersections. The consideration of these matrices is the subject of the next section.

2.1 From Berger graphs and Dynkin diagrams to Berger matrices

In the previous section we have established the existence of Dynkin-like graphs, possibly not corresponding to any of the known Lie or affine Kac-Moody algebras. The information contained in the graphs can be encoded in a more workable structure: a matrix of integer numbers to be defined. If these “Dynkin” graphs are somehow related to possible generalizations of the Lie and affine Kac-Moody algebra concepts, it is then natural to look for possible generalizations of the corresponding affine Kac-Moody Cartan matrices when searching for possible ways of assigning integral matrices to them.

One possibility which could serve us of guide is to suppose that this affine property remains: matrices with determinant equal to zero and all principal minors positive. We will see in what follows that this is a sensitive choice, on the other hand it turn out that the usual conditions on the value of the diagonal elements has to be abandoned.

(*Building Matrices from graphs.*) We assign to any generalized Dynkin diagram, a set of vertices and lines connecting them, a matrix, B , whose non diagonal elements are either zero or are negative integers. There are different possibilities, for non diagonal elements, considering for the moment the most simple case of “laced” graphs leading to symmetric assignments, we have:

- there is no line from the vertex i to the vertex j . In this case the element of the matrix $B_{ij} = 0$
- there is a single line connecting $i - j$ vertices. In this case $B_{ij} = -1$

The diagonal entries should be defined in addition. As a first step, no special restriction is applied and any positive integer is allowed. We see however that very quickly only a few possibilities are naturally selected. The diagonal elements of the matrix are two for CY2 originated graphs but are allowed to take increasing integer numbers with the dimensionality of the space, 3, 4... for $CY_{3,4,\dots}$.

We have checked (see also Ref.[9]) a large number of graphs and matrices associated to them, obtained by inspection considering different possibilities. Some regularities are quickly disclosed. In first place it is easy to see that there are graphs where the number of lines outgoing a determined vertex can be bigger than two, in cases of interest they will be 3, graphs from CY3, or bigger in the cases of graphs coming from CY4 and higher dimensional spaces. Some other important regularities appear. The matrices are genuine generalizations of affine matrices. Their determinant can be made equal to zero and all their principal minors made positive by careful choice of the diagonal entries depending on the Calabi-Yau dimension and n-ary structure.

Moreover, we can go back to the defining reflexive polyhedra and define other quantities in purely geometrical terms. For example we can consider the position or distance of each of the vertices of the generalized Dynkin diagram to the intersecting reflexive polyhedra. Indeed, it has been remarked [5] that Coxeter labels for affine Kac-Moody algebras can be obtained directly from the graphs: they correspond precisely to this “distance” between individual nodes and some defined intersection which separates “left” and “right” graphs.

Intriguingly, this procedure can be easily generalized to our case, one can see that, by a careful choice of the entry assignment for the corresponding matrix, it follows Coxeter labels can be given in a proper way: they have the expected property of corresponding to the elements of the null vector a generalized Cartan matrix.

From the emerging pattern of these regularities, we are lead to define a new set of matrices, generalization of Cartan matrices in purely algebraic terms, the Berger, or Berger-Cartan-Coxeter matrices. This will be done in the next paragraph.

(*Definition of affine Berger Matrices*). Based on previous considerations, we define, now in purely algebraic terms, the so called Berger Matrices. We suggest the following rules for them, in what follows we will see step by step how they lead to a consistent construction generalizing the Affine Kac-Moody concept. A Berger matrix is a finite integral matrix characterized by the following data:

$$\begin{aligned}\mathbb{B}_{ii} &= 2, 3, 4.. \\ \mathbb{B}_{ij} &\leq 0, \quad \mathbb{B}_{ij} \in \mathbb{Z}, \\ \mathbb{B}_{ij} = 0 &\mapsto \mathbb{B}_{ji} = 0, \\ \text{Det } \mathbb{B} &= 0, \\ \text{Det } \mathbb{B}_{\{(i)\}} &> 0.\end{aligned}$$

The last two restrictions, the zero determinant and the positivity of all principal proper minors, corresponds to the *affine condition*. They are shared by Kac-Moody Cartan matrices, so we expect that the basic definitions and properties of those can be easily generalized. However, with respect to them, we relaxed the restriction on the diagonal elements. Note that, more than one type of diagonal entry is allowed: 2, 3, .. diagonal entries can coexist in a given matrix. This apparently minor modification has in turn important consequences [17] when we define a Weyl group for the theory.

For the sake of convenience, we define also “*non-affine*” *Berger Matrices* where the condition of non-zero determinant is again imposed. These matrices does not seem to appear naturally resulting from polyhedron graphs but they are useful when defining root systems and Weyl group for the affine case by extension of them. They could play the same role of basic simple blocks as finite Lie algebras play for the case of affine Kac-Moody algebras.

The important fact to be remarked here is that this definition lead us to a construction with the right properties we would expect from a generalization of the Cartan matrix idea.

(*From any Berger matrix to a Berger-Dynkin diagram*.) The systematic enumeration of the various possibilities concerning the large family of possible Berger matrices can be facilitated by the introduction for each matrix of its generalized Dynkin diagram. As we intend that the definition of this family of matrices be independent of algebraic geometry concepts we need an independent definition of these diagrams. Obviously the procedure given before can be reversed to allow the deduction of the generalized Dynkin diagram from its generalized Cartan or Berger Matrix. An schematic prescription for the most simple cases could be: A) For a matrix of dimension n , define n vertices and draw them as small circles. In case of appearance of vertices with different diagonal entries, some graphical

distinction will be performed. Consider all the element i, j of the matrix in turn. B) Draw one line from vertex i to vertex j if the corresponding element A_{ij} is non zero.

In what follows, we show that indeed these kind of matrices and Dynkin diagrams, exist beyond those purely defined from Calabi-Yau newton reflexive polyhedra. In fact we show that there are infinity families of them where suggestive regularities appear.

It seems easy to conjecture that the set of all, known or generalized, Dynkin diagrams obtained from Calabi-Yau spaces can be described by this set of Berger matrices. It is however not so clear the validity of the opposite question, whether or not the infinite set of generalized Dynkin diagrams previously defined can be found digging in the Calabi-Yau (n, a) structure indicated by UCYA. For physical applications however it could be important the following remark. Theory of Kac-Moody algebras show us that for any *finite* or *affine* Kac-Moody algebra, every proper subdiagram (defined as that part of the generalized Coxeter-Dynkin diagram obtained by removing one or more vertices and the lines attached to these vertices) is a collection of diagrams corresponding to *finite* Kac-Moody algebras. In our case we have more flexibility. Proper subdiagrams, obtained eliminating internal nodes or vertices, are in general collections of Berger-Coxeter-Dynkin diagrams corresponding to other (affine by construction)Berger diagrams *or* to *affine* Kac-Moody algebras. This property might open the way to the consideration of non-trivial extensions of SM and string symmetries.

(*The Berger Matrix as an inner product: construction of root spaces.*) For further progress, the interpretation of a Berger matrix as the matrix of divisor intersections $B_{ij} \sim D_i \cdot D_j$ in Toric geometry could be useful for the study of the viability of fans of points associated to them, singularity blow-up, and the existence of Calabi-Yau varieties itself. This geometrical approach will be pursued somewhere else [17]. However, for algebraic applications, and with the extension of the CLA and KMA concepts in mind, the interpretation of these matrices as matrices corresponding to a inner product in some vector space is most natural which is our objective now.

The Berger matrices are obtained by weakening the conditions on the generalized Cartan matrix \hat{A} appearing in affine Kac-Moody algebras. In what concern algebraic properties, there are no changes, it remains intact the condition of semi-definite positiveness, this allows to translate trivially many of the basic ideas and terminology for roots and root subspaces for appearing in Kac-Moody algebras. Clearly, the problem of expressing the “simple” roots in a orthonormal basis was an important step in the classification of semisimple Cartan-Lie algebras.

The apparent minor modification which has been introduced: to allow for diagonal entries different from two, has however deep impact in another place: when we define the generalization of the Weyl group of transformations, we easily realize that this is in general an infinite group enlarged, with respect the also infinite Weyl group appearing in affine Kac-Moody case, by a new infinite set of transformations. The detailed study of the properties of these set of transformations will be the object of a next study elsewhere [17].

For a Berger matrix B_{ij} of dimension n , the rank is $r = n - 1$. The $(r + 1) \times (r + 1)$ dimensional is nothing else that a generalized Cartan matrix. We can suppose without loss of generality this matrix to be symmetric (if we start with a non-symmetric Cartan

matrices, as actually can happen, we can always apply some symetrization process as it can be done with Kac-Moody algebras, see for example Ref.[18] for a detailed explanation). We expect that a simple root system $\Delta^0 = \{\alpha_1, \dots, \alpha_r\}$ and an extended root system by $\hat{\Delta}^0 = \alpha_0, \alpha_1, \dots, \alpha_r$, can be constructed. The defining relation is that the (scaled) inner product of the roots is

$$\alpha_i \cdot \alpha_j = \hat{\mathbb{B}}_{ij} \quad 1 \leq i, j \leq n. \quad (2.1)$$

The set of roots α_i are the simple roots upon which our generalized Cartan Matrix is based. They are supposed to play the analogue of a root basis of a semisimple Lie Algebra or of a Kac-Moody algebra. Note that, as happens in KMA Cartan matrices, for having the linearly independent set of α_i vectors, we generically define them in, at least, a $2n - r$ dimensional space H . In our case, as $r = n - 1$, we would need a $n + 1$ dimensional space. Therefore, the set of n roots satisfying the conditions above has to be completed by some additional vector, the “null root”, to obtain a basis for H . The consideration of these complete set of roots will appear in detail elsewhere [17].

A generic root, α , has the form

$$\alpha = \sum_i c_i \alpha_i$$

where the set of the coefficients c_i are either all non-negative integers or all non-positive integers. In this $n + 1$ dimensional space H , generic roots can be defined and the same generalized definition for the inner product of two generic roots α, β as in affine Kac-Moody algebras applies. This generalized definition reduces to the inner product above for any two simple roots.

The roots can be of the “same length” or not. A difference with respect KMA is that the condition of having all roots the same length, symmetry of the matrix and of being “simple laced” are not equivalent now. This is a simple consequence of allowing more than one type of diagonal entries in the matrix.

Since B is of rank $r = n - 1$, we can find one, and only one, non zero vector μ such that

$$B\mu = 0.$$

The numbers, a_i , components of the vector μ , are called Coxeter labels. The sums of the Coxeter labels $h = \sum \mu_i$ is the Coxeter number. For a symmetric generalized Cartan matrix only this type of Coxeter number appear.

3. Some simple examples: from graphs to roots

Let consider the reflexive polyhedron, which corresponds to a K3-sliced CY_3 space and which is defined by two extended vectors [9] $\vec{k}_L^{ext}, \vec{k}_R^{ext}$, one coming from the following set of vectors $S_L = \{(0, 0, 0, 0, \vec{k}_1), (0, 0, 0, 0, \vec{k}_2), (0, 0, \vec{k}_3), \dots (perms) \dots\}$, where the remaining dots correspond to permutations of the position of zeroes and vectors k , for example permutations of the type $\{(0, k, 0, 0, 0), (k, 0, 0, 0, 0), \text{ etc } \}$. The other one comes from the set

$S_R = \{(0, \vec{k}_4), \dots (perms)..\}$, The vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3$ are respectively any of the five RWVs of dimension 1,2 and 3. The vector \vec{k}_4 correspond to any of the 95 $K3$ RWVs of dimension four.

Let consider, as a simple example, a quintic CY_3 which its reflexive polyhedron which is defined by two extended vectors, $\vec{k}_{1L}^{(ext)} = (1, 0, 0, 0, 0)$ and $\vec{k}_{2R}^{(ext)} = (0, 1, 1, 1, 1)$ (which correspond to the choice $\vec{k}_4 = (1, 1, 1, 1)$).

Note that the global slice structure of the CY_3 corresponding to the weight vector $\vec{k}_5 \equiv \vec{k}^{(ext)} = \vec{k}^{(ext)}_{1L} + \vec{k}^{(ext)}_{2R} = (1, 1, 1, 1, 1)$ is determined by *both* extended vectors appearing in its sum.

The top and bottom, or left and right, skeletons of reflexive polyhedron are determined by extended vectors, $\vec{k}_{1L}^{(ext)}, \vec{k}_{2R}^{(ext)}$ respectively. The left skeleton will be a tetrahedron with 4-vertices, 6 edges and a number of internal points over the edges as indicated in the Figure 3 of Ref.[9].

In the next sections, we will consider in turn any of the cases corresponding to the vectors of different dimension enumerated above. We will see how everycase will give us a rich variety of algebraic generalizations of the Cartan matrices and root systems appearing in Kac-Moody algebras.

3.1 Graph from vector (1)+(1111): Construction of the root systems

The graph associated to vector $\vec{k}_L = (1)$, (or (10000), including all zero components), and assuming $\vec{k}_R = (1111)$ is a closed graph, a tetrahedron (see Ref.[9] for other type of graphs when combining with other \vec{k}_R vectors). Following the rules given above we can form the associated Berger matrix, or generalized Cartan matrix. This matrix is a 28×28 matrix corresponding to four vertices and 24 internal nodes over the six edges. One can actually check that this matrix is of affine type. Inspired by this graph we define a full family of graphs where the number of internal nodes over the edges are altered. Let us focus in the most simple case, where the all internal nodes are eliminated and we are left with only the four vertices (figure 1). From this simplified graph, not directly obtained from the CY polyhedra, we built the corresponding matrix according to the rules given in the previous section, we will call this a $CY\mathbb{B}_3^{(1)}$ matrix. Let us also consider for the sake of comparison the graphs and Cartan matrices for the affine algebra $A_2^{(1)}$. The associated graph, a standard Coxeter-Dynkin diagram, is just one of the triangles corresponding to the faces of this simple tetrahedron. The well known Cartan matrix for the $A_2^{(1)}$ affine algebra and our Berger matrix $CY3B_3^{(1)}$ are respectively.

$$A_2^{(1)} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad CY3B_3^{(1)} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \quad (3.1)$$

As the next step, we build a system of vectors, the equivalent of a root system, which satisfy the Berger matrix interpreted as an inner product. Let us remind first that for the

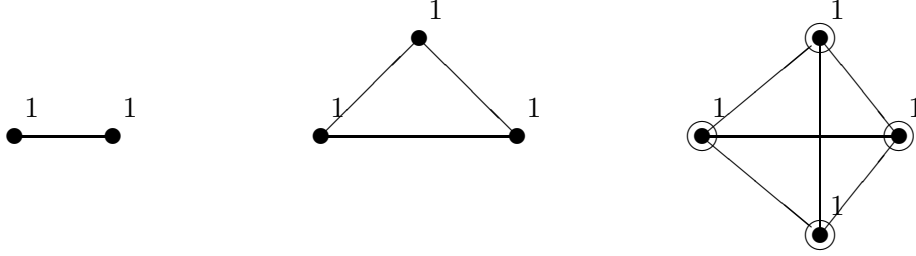


Figure 1: Respectively from left to right, the Dynkin diagrams of $A_1, A_2^{(1)}$ and the generalized Berger-Dynkin diagram tetrahedron dyagram $CY3B_3$. Coxeter Labels are attached at the nodes. Any graph of the series is an edge or face of the following graph. In this and following graphs double-circled nodes yield diagonal entries equal to three in the corresponding matrix.

well known affine case $A_2^{(1)}(n = 3, r = 1)$, supposing a set of orthonormal canonical basis $\{e_i\}$, one obtains three vectors α_i such as:

$$\begin{aligned}\alpha_1 &= e_1 - e_2, \\ \alpha_2 &= e_2 - e_3, \\ \alpha_3 &= e_3 - e_1,\end{aligned}$$

the root $\alpha_0 \equiv -\alpha_3$ is the eldest root vector in the A_2 algebra. For the Cartan matrix of the affine Kac-Moody algebra, $A_2^{(1)}$, the sum of all simple roots with certain coefficients, the Coxeter labels, is equal zero. Diagonalizing and obtaining the zero mode of the matrix, one can easily see that for the full $A_r^{(1)}$ series all the Coxeter labels are equal to one (see for example [15, 16]), we have:

$$1 \cdot \alpha_1 + 1 \cdot \alpha_2 + 1 \cdot \alpha_3 = 0. \quad (3.2)$$

For the case corresponding to $CYB_3^{(1)}$, a symmetric $n = 4, r = 3$ matrix, we obtain the following simple root system. As we mentioned before we need a vector space of dimension $d > 2n - r = 5$ to express the roots, we use in this example an overcomplete 6 dimensional orthonormal basis of vectors $(\{e_i\}, i = 1, \dots, 6)$. The advantage of this basis, among many other choices, being that the coefficients of the vectors are specially simple here:

$$\begin{aligned}\alpha_1 &= e_1 + e_2 + e_3, \\ \alpha_2 &= -e_1 - e_4 + e_5\end{aligned}$$

$$\alpha_3 = -e_2 + e_4 + e_6,$$

$$\alpha_4 = -e_3 - e_5 - e_6,$$

It is easy to check that these simple roots satisfy all the restrictions imposed by the Berger matrix $\alpha_i \cdot \alpha_j = B_{ij}$. Following the nomenclature of Kac-Moody algebras we say that all roots in this case are of the same length: a property of being “simple laced”. We will see however that the concepts of simply laced, symmetric matrices and equal-length roots are not totally equivalent for Berger matrices and should be distinguished.

The diagonalization of the matrix gives us the zero mode vector, $B\mu = 0$. In this case $\mu = (1, 1, 1, 1)$ and $h = 4$. The affine condition satisfied by the set of simple roots:

$$1 \cdot \alpha_1 + 1 \cdot \alpha_2 + 1 \cdot \alpha_3 + 1 \cdot \alpha_4 = 0, \quad (3.3)$$

the coefficients of this vector being the Coxeter labels, components of the vector μ .

The Berger matrix as $CY3B3$ and its tetrahedron graph can be easily generalized to any dimension. We can consider an infinite series of matrices and hyper-tetrahedron graphs originated from CY_n spaces defined as those $(n+1) \times (n+1)$ matrices with all non-diagonal entries equal to -1 and n in the diagonal. For illustration, the matrix corresponding to $CY4\mathbb{B}_4^{(1)}$ is

$$CY4\mathbb{B}_4^{(1)} = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}, \quad (3.4)$$

the determinant of this matrix is effectively zero $\det(CY4\mathbb{B}_4^{(1)}) = 0$.

For any dimension n it is straightforward to show that $\det(-\mathbf{1} + x I) = (x - n)x^{n-1}$ where $\mathbf{1}, I$ are respectively the matrix filled with 1 and the identity matrix. Thus, if we choose for the diagonal entries the value $x = n$, the full family of matrices $CYnB$ acquires zero determinant. According to the previous formula for the determinant, this choice is *unique*. It is also straightforward to show that for the full family of matrices we have the vector of Coxeter labels $\mu_{(n)} = (1, ..(n)..1)$ and the affine condition $\sum_n \alpha_i = 0$.

3.1.1 Additional algebraically defined Berger matrices of any order.

In the previous paragraphs, we have used the definition of the Berger graphs and matrices independently from their Calabi-Yau polyhedra origin, we want to extend this single matrix to a infinite series of related matrices. On purely algebraic grounds we consider a series of new Berger graphs and matrices that we will denote as $CY3B_s^{(1)}$, $s \geq 3$. The graphs are obtained including any number of internal Cartan nodes on one or more than one of the edges of the defining tetrahedron considered in the previous section. An arbitrary example of a tetrahedron graph with a number of internal nodes is given in Fig.(2). We consider this graph as a generalization of affine Kac-Moody algebras: each face of the tetrahedron resembles a $A_r^{(1)}$ triangle graph, where $r + 1$ internal nodes appear.

From this enlarged graph we build corresponding Berger matrices using the rules given previously. These matrices will differ from the basic $CY\mathbb{B}_3^{(1)}$ matrix above by the appearance of $s - 3$ elements equal to two in the diagonal, corresponding to the new internal nodes. The diagonal elements corresponding to the unchanged vertex nodes remain the same.

The new matrices thus built are “well-behaved” Berger matrices: The algebraic Berger conditions are fulfilled for any number of edge nodes. Thus we have an infinite series of matrices in parallel to what happens in the Cartan-Lie case with the series $A_r^{(1)}$. Let us note however that for this extension, no reference to Calabi-Yau geometry has been used.

As an illustrating example, let us write thee cases where $s = 4, 5$, we include one or two new internal nodes respectively in one of the edges of the tetrahedron. The the Berger matrices are

$$CY3\mathbb{B}_4^{(1)} = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 3 & -1 & -1 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 3 \end{pmatrix}, \quad CY3\mathbb{B}_5^{(1)} = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & -1 & 3 \end{pmatrix}. \quad (3.5)$$

One can compute explicitly that the determinant of these matrices is equal to zero and that all the principal proper minors are positive. Clearly the position of the diagonal elements is unimportant and can be changed by relabeling of the nodes.

We proceed now to obtain a system of simple roots, we show just as illustration the case corresponding to the second matrix, $CY3\mathbb{B}_5^{(1)}$. One needs in this case a vector space of dimension $d \geq 2n - r = 7$, where $r = 1$, to express the roots, we use again an overcomplete set $(\{e_i\}, i = 1, \dots, 8$ where the coefficients appears specially simple:

$$\begin{aligned} \alpha_1 &= e_1 + e_2 + e_3, \\ \alpha_2 &= -e_1 + e_7, \\ \alpha_3 &= -e_7 + e_8 \\ \alpha_4 &= -e_8 - e_4 + e_5, \\ \alpha_5 &= -e_2 + e_4 + e_6, \\ \alpha_6 &= -e_3 - e_5 - e_6. \end{aligned}$$

The root system contains now two additional binary roots α_2, α_3 . The roots are not of the “same length”: as we have mentioned before the condition of having all roots the same length, symmetry of the matrix and of being “simple laced” are not equivalent now. This is a simple consequence of allowing more than one type of diagonal entries in the matrix. The Coxeter labels and affine condition on the roots are easily obtained. The diagonalization of the matrix gives us the zero mode vector, $B\mu = 0$. In this case $\mu = (1, 1, 1, 1, 1, 1)$ and $h = 6$. The affine condition satisfied by the set of simple roots: $\sum_6 \alpha_6 = 0$ the coefficients of this vector being the Coxeter labels, components of the vector μ .

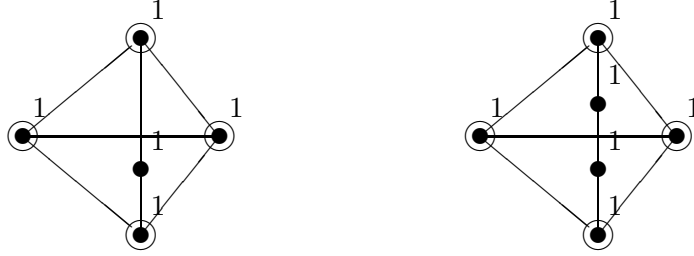


Figure 2: Generalized Berger-Dynkin tetrahedron dyagram CYB_3 including one or two additional internal nodes in one of the edges.

3.2 The left (111)-right (1111) case.

The graph associated to the case where we have a left vector (111) plus a right vector (1111) as before, can equally be extracted from the reflexive Newton polyhedron following a simple procedure. The result appears in Fig.(3). In this case, we directly obtain the Coxeter-Dynkin diagram corresponding to the affine algebra $E_6^{(1)}$. We can easily check that following the rules given above we can form an associated Berger matrix, which, coincides with the corresponding generalized Cartan matrix of the the affine algebra $E_6^{(1)}$. The well known Cartan matrix for this is:

$$E_6^{(1)} = CYB3 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 0 & -1 & 2 \end{pmatrix} \quad (3.6)$$

The root system is well known, we have (in a, minimal, ortonormal basis $(\{e_i\}, i = 1, ..., 8)$):

$$\begin{aligned} \alpha_1 &= -\frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 - e_8) \\ \alpha_2 &= (e_2 - e_1) \\ \alpha_3 &= (e_4 - e_3) \\ \alpha_4 &= (e_5 - e_4) \end{aligned}$$

$$\begin{aligned}
\alpha_5 &= (e_1 + e_2) \\
\alpha_6 &= -\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8) \\
\alpha_7 &= -e_2 + e_3
\end{aligned}$$

Coxeter labels and affine condition are easily reobtained. The diagonalization of the matrix gives us the zero mode vector, $B\mu = 0$. In this case the Coxeter labels are $\mu = (1, 2, 1, 2, 1, 2, 3)$ and $h = 12$. The affine condition satisfied by the set of simple roots is also well known

$$\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + 3\alpha_7 = 0$$

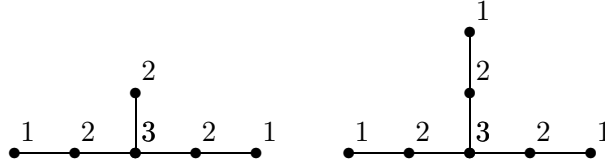


Figure 3: Dynkin diagrams for the exceptional E_6 and affine $E_6^{(1)}$ algebras (left and right respectively).

Now, we ask the question on how we can generalize the appearance this affine algebra using our Berger construction. We can consider at least two different ways.

In the first way we will consider the standard graph and matrix corresponding to $E_6^{(1)}$ and attach an increasing number of legs to the central node. The number of nodes in each leg is a well defined number: equals the number of legs itself minus one. For example in the $E_6^{(1)}$ case, associated to vector $(111)[3]$ we have three legs with two nodes in each leg, successive diagrams will include 4,5.. legs with 3,4.. nodes. Loosely speaking, we say that these Dynkin graphs are associated to the series of vectors $(1, \dots, 1)[n]$ for any n . In the second way, we will start again from the standard graph and matrix corresponding to $E_6^{(1)}$ and insert additional internal nodes. Let us note that one can show that internal nodes can be added or appended to the legs of the graph, this would be already known extensions of the affine algebra (see for example, Ref.[14]). In our context, this kind of extension would be just an example of the addition of internal nodes as we did in the previous section, similarly we could see that we can draw graphs and obtain matrices in a consistent way.

But our procedure allow us for more complicated extensions: we can add not only just nodes to the legs, but legs itself. In this way we are able to glue together in a non-trivial way Coxeter-Dynkin diagrams corresponding to different affine algebras and obtain Berger(-Coxeter-Dynkin) matrices consistent with the definitions given above.

Surprisingly, these two ways of generalization seem to be mutually exclusive. No example is known to us where both can be simultaneously performed (see discussion at the end of section 3.3).

As we have mentioned before, we somehow generalize the decomposition of Kac-Moody algebras. Theory of Kac-Moody algebras show us that for any *finite* or *affine* Kac-Moody algebra, every proper subdiagram (defined as that part of the generalized Coxeter-Dynkin diagram obtained by removing one or more vertices and the lines attached to these vertices) is a collection of diagrams corresponding to *finite* Kac-Moody algebras. In our case we have more flexibility. Proper subdiagrams, obtained eliminating internal nodes or vertices, are in general collections of Berger-Coxeter-Dynkin diagrams corresponding to other (affine by construction)Berger diagrams *or* to *affine* Kac-Moody algebras. This property clearly paves the way to the consideration of non-trivial extensions of SM and string symmetries.

In the next two sections, examples of each of the two procedure outlined above will be treated in turn.

3.3 An “exceptional” series of graphs associated to $(1, \dots, 1)$

We discuss the first way of generalization explained above. One takes the standard graph and matrix corresponding to $E_6^{(1)}$ and attaches an increasing number of legs to the central node. The number of nodes in each leg is a well defined number: equals the number of legs itself minus one. For example in the $E_6^{(1)}$ case, associated to vector $(111)[3]$ we have three legs with two nodes in each leg, successive diagrams will include 4,5.. legs with 3,4.. nodes. Loosely speaking, we say that these Dynkin graphs are associated to the series of vectors $(1, \dots, 1)[n]$ for any n . The graph associated to vector (1111) appears in Fig.4 (center). This one have much more different structure comparing to the previous Berger graphs. Its more important characteristic it is that it has a vertex-node with outgoing 4-lines. We call this graph a generalized $E_6^{(1)}$ graph of type I.

The Berger matrix is obtained from the planar graph according to the standard rules. We assign different values (2 or 3) to diagonal entries depending if they are associated to standard nodes or to the central vertex. The result is the following 13×13 symmetric matrix containing, as more significant difference, an additional 3 diagonal entry:

$$CY3B_0 = \left(\begin{array}{ccc|ccc|ccc|ccc|c} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ \hline -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 3 \end{array} \right) \quad (3.7)$$

One can check that this matrix fulfills the conditions for Berger matrices. Its determinant is zero while the rank $r = 12$. All the principal minors are positive.

One can obtain a system of roots $(\alpha_i, i = 1, \dots, 13)$ in a orthonormal basis. Considering the orthonormal canonical basis $(\{e_i\}, i = 1, \dots, 12)$, we obtain:

$$\begin{aligned}
\alpha_1 &= -(e_1 - e_2) \\
\alpha_2 &= \frac{1}{2}[(e_1 - e_2 - e_3 + e_4 + e_5 + e_6) + (e_8 - e_7)] \\
\alpha_3 &= -(e_8 - e_7) \\
\alpha_4 &= (e_4 - e_3) \\
\alpha_5 &= (e_5 - e_4) \\
\alpha_6 &= (e_6 - e_5) \\
\alpha_7 &= (e_1 + e_2) \\
\alpha_8 &= -\frac{1}{2}[(e_1 + e_2 - e_9 - e_{10} - e_{11} - e_{12}) + (e_8 + e_7)] \\
\alpha_9 &= (e_8 + e_7) \\
\alpha_{10} &= -(e_{10} - e_9) \\
\alpha_{11} &= -(e_{11} - e_{10}) \\
\alpha_{12} &= -(e_{12} - e_{11}) \\
\alpha_{13} &= e_3 - e_2 - e_9
\end{aligned}$$

The assignment of roots to the nodes of the Berger-Dynkin graph is given in Fig.5. It easily to check the inner product of these simple roots leads to the Berger Matrix $a_i \cdot a_j = B_{ij}$. This matrix has one null eigenvector, with coordinates, in the α basis, $\mu = (3, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 1, 4)$. The Coxeter number is $h = 22$. One can check that these Coxeter labels are identical to those obtained from the geometrical construction [5, 9]. They are shown explicitly in Fig.4. Correspondingly the following linear combination of the roots satisfies the affine condition:

$$4\alpha_0 + 3\alpha_1 + 2\alpha_2 + \alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 3\alpha_7 + 2\alpha_8 + \alpha_9 + 3\alpha_{10} + 2\alpha_{11} + \alpha_{12} = 0$$

One could try to pursue the generalization process of graphs and matrices adding internal nodes to this case as it has been done previously. Surprisingly, in contradiction to previous case where an infinite series of new graphs and matrices can be obtained, this is however and “exceptional” case. No infinite series of graphs can be obtained in this way.

However, we can suppose that the graphs corresponding to the vectors $(111), (1111), (1, \dots, n, 1)$ (see Fig.4), generate an infinite series of new Cartan Matrices $CY_{2,3,4,\dots}E_6^{(1)}$ ordered with respect to Calabi-Yau space dimension. The first term in this infinite series starting from the $E_6^{(1)}$ case. All our three restrictions to the Berger graphs can be easily checked. The determinant of these matrices for all cases is equal zero.

Similarly, one can easily find [9] the dimensional generalizations of the $E_7^{[1]}$ and $E_8^{[1]}$ graphs (corresponding to the choice of vectors $(112), (1123)$), “exceptional” cases themselves

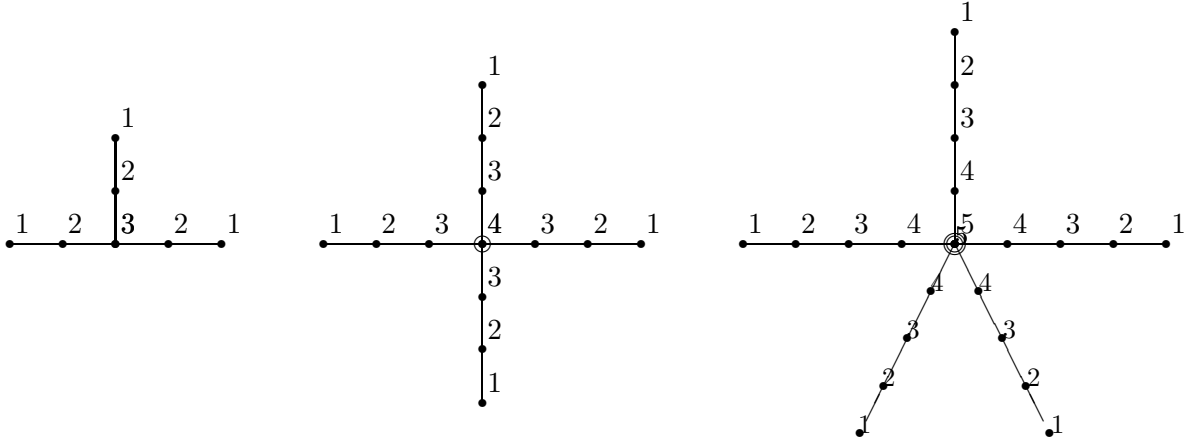


Figure 4: Berger-Dynkin diagrams for the affine $E_6^{(1)}$ and its generalizations $CY3 - E_6^{(1)}$ and $CY4 - E_6^{(1)}$ diagrams (left and right respectively).

in the same sense, which give us the infinite series linked with the dimension of d . A complete exposition of these cases will appear in Ref.[17].

Let us finish this illustrative example mentioning the possibility of recovering the Coxeter-Dynkin diagrams products of finite Lie algebras or affine Kac-Moody algebras by eliminating one the roots appearing above. It is apparent from the example shown in Fig.6 that they reappear as proper subdiagrams when removing one or more vertices and the lines attached to these vertices.

3.4 Another generalization: the (111) graph plus internal nodes

As a new example, in this section we will follow a different way of generalization of the $E_6^{(1)}$ graph on pure algebraic ways, independently of Calabi-Yau classification arguments. For this purpose, we take as starting point the Berger graph $E_6^{(1)}$ corresponding to the RWV vector (111), shown in Figs.3 or 4, and we add a fourth leg to the central node. At the end of the new leg we will attach a new copy of a $E_6^{(1)}$ Coxeter-Dynkin diagram. The length of this internal leg connecting the two affine algebra copies is arbitrary.

The important fact of this seemingly arbitrary construction is that if we write the corresponding matrix, according to the rules Eqs.2.1, we obtain a new Berger matrix, that is, a matrix which fulfills the same defining conditions. In particular the affine condition and the positivity conditions. This fact lead us again to consider the importance on its own of the Berger matrix definition as a generalization of the Kac-Moody Cartan matrix. Thus we are lead to consider the series of Berger graphs and matrices built according to this algorithm as infinite: in an obvious symbolic notation $E_6^{(1)} + A_r^{(1)} + E_6^{(1)}$ where r is arbitrary.

As an example we draw in Fig. 6 the graph resulting from the incorporation of a two node extra leg. The total number of nodes, in obvious notation, is $7_L + 2_{int} + 7_R = 16$. The 16×16 Berger matrix for this graph is the following.

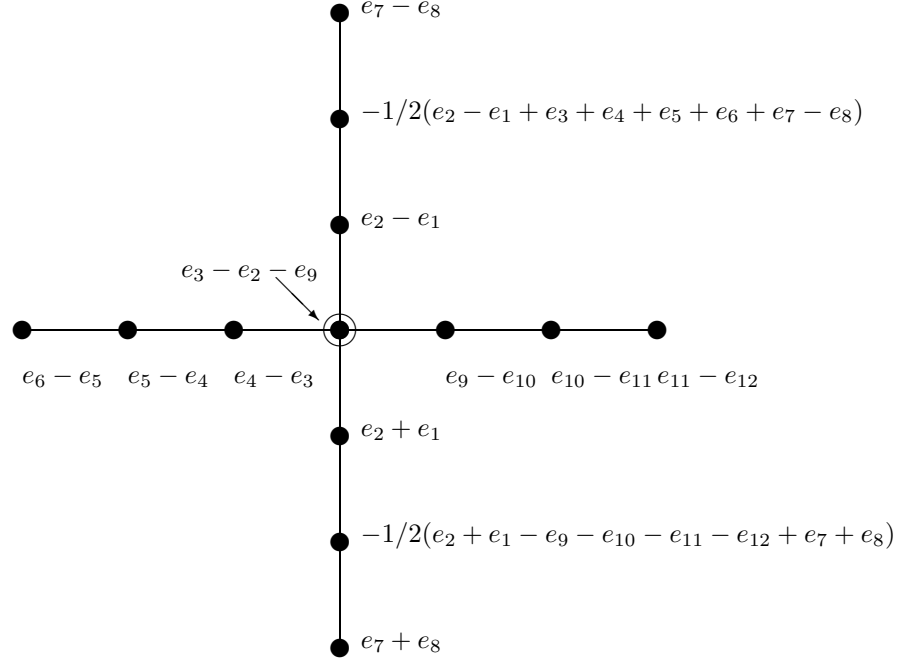


Figure 5: Berger-Dynkin diagram and root system for the $CY3 - E_6^{(1)}$ matrix.

$$CY3B = \left(\begin{array}{cccccc|cccccccc} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \end{array} \right)$$

One can convince oneself that the determinant of this matrix is equal zero and the principal minors are positive definite: The Berger matrix can be considered of built from three diagonal blocks, distinguished by lines, containing respectively parts which partially re-

semble (in fact they only differ by the apparition of 3's in the joining nodes) $E_6^{(1)}$ and A_7 Cartan matrix copies.

Let construct for this graph the root system consisting from the the $7_L + 2_{int} + 7_R = 16$ nodes (see Fig. 6). The 16 roots are denoted as $\{\alpha_i, \dots, \hat{\alpha}_1, \hat{\alpha}_2, \tilde{\alpha}_i, \dots\}, i = 1 - 7$. In a similar way the $16 + 3$ dimensional orthonormal canonical basis is divided in three sectors and given by $\{e_i, \dots, \hat{e}_1, \hat{e}_0, \hat{e}_{-1}, \tilde{e}_i, \dots\}, i = 1 - 8$.

$$\begin{aligned}
\alpha_1 &= -\frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 - e_8) \\
\alpha_2 &= (e_2 - e_1) \\
\alpha_3 &= (e_4 - e_3) \\
\alpha_4 &= (e_5 - e_4) \\
\alpha_5 &= (e_1 + e_2) \\
\alpha_6 &= -\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8) \\
\alpha_7 &= -e_2 + e_3 - \hat{e}_1 \\
\hat{\alpha}_1 &= \hat{e}_1 - \hat{e}_0 \\
\hat{\alpha}_2 &= \hat{e}_0 - \hat{e}_{-1} \\
\tilde{\alpha}_7 &= -\tilde{e}_2 + \tilde{e}_3 - \hat{e}_{-1} \\
\tilde{\alpha}_6 &= -\frac{1}{2}(\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 + \tilde{e}_4 + \tilde{e}_5 - \tilde{e}_6 - \tilde{e}_7 + \tilde{e}_8) \\
\tilde{\alpha}_5 &= (\tilde{e}_1 + \tilde{e}_2) \\
\tilde{\alpha}_4 &= (\tilde{e}_5 - \tilde{e}_4) \\
\tilde{\alpha}_3 &= (\tilde{e}_4 - \tilde{e}_3) \\
\tilde{\alpha}_2 &= (\tilde{e}_2 - \tilde{e}_1) \\
\tilde{\alpha}_1 &= -\frac{1}{2}(-\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 + \tilde{e}_4 + \tilde{e}_5 + \tilde{e}_6 + \tilde{e}_7 - \tilde{e}_8)
\end{aligned}$$

The matrix has a null eigenvector with entries, the generalized Coxeter labels,

$$\mu = (1, 2, 1, 2, 1, 2, 3, 3, 3, 3, 2, 1, 2, 1, 2, 1).$$

These numbers are shown attached to the nodes of the generalized Dynkin graph in Fig.6. The Coxeter number is $h = 30$. Note that the Coxeter labels corresponding to one sector $E_6^{(1)}$ are precisely a subset of those $= (1, 2, 1, 2, 1, 2, 3)$. The following linear combination of the roots satisfies the affine condition:

$$\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + 3\alpha_7 + 3\hat{\alpha}_1 + \hat{\alpha}_2 + \tilde{\alpha}_1 + 2\tilde{\alpha}_2 + \tilde{\alpha}_3 + 2\tilde{\alpha}_4 + \tilde{\alpha}_5 + 2\tilde{\alpha}_6 + 3\tilde{\alpha}_7 = 0.$$

Note that, as it could not be otherwise, the affine condition entangles all the vectors of the system: there is only one affine condition, the individual affine conditions corresponding to the individual sectors are not satisfied.

Generalizations of the E_7^1 and E_8^1 series of graphs, corresponding respectively to vectors (112) and (113) can also be easily obtained [17]. It is suggestive to consider the procedure

outlined in this section as a new way of unifying two semisimple algebras within a larger algebra. By the reverse process, the decomposition of the Berger diagram leads to proper, Kac-Moody, subdiagrams and the reduction of the Berger matrix to Kac-Moody blocks. This can be performed eliminating the nodes corresponding to one or more roots and the lines attached to them.

In particular, it is suggestive to consider the case where a Berger matrix resembling by blocks (procedure exemplified in Fig.6)

$$G \sim CY3B(8_L + 2_{int} + 8_R) \sim E_7^{(1)} + A_r + E_7^{(1)}$$

or

$$G \sim CY3B(9_L + 2_{int} + 9_R) \sim E_8^{(1)} + A_r + E_8^{(1)}$$

is reduced to standard blocks and where the one affine condition or central vector is broken down to three different affine conditions for each of the blocks. We could compare this case with the heterotic string example where the unified gauge group is obtained by standard direct sum of algebras $G = E_8 + E_8$.

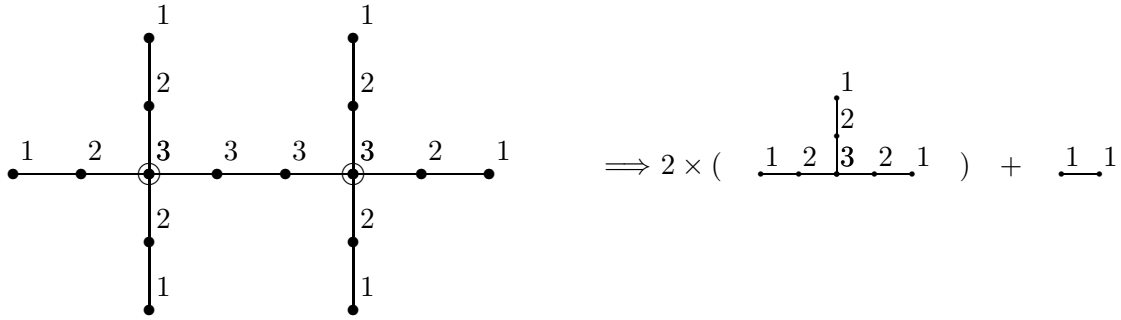


Figure 6: Berger-Dynkin diagrams for $CY3B(7 + 2 + 7)$ diagram and breaking down into two copies of affine $E_6^{(1)}$ diagrams.

4. Summary, additional comments and conclusions

The interest to look for new algebras beyond Lie algebras started from the $SU(2)$ -conformal theories (see for example [19, 20]). One can think that geometrical concepts, in particular algebraic geometry, could be a natural and more promising way to do this. This marriage of algebra and geometry has been useful in both ways. Let us remind that to prove mirror symmetry of Calabi-Yau spaces, the greatest progress was reached with using the technics of Newton reflexive polyhedra in [4].

We considered here examples of graphs from $K3 = CY_2$ and new graphs from CY_d ($d \geq 3$) reflexive polyhedra, which could be in one-to-one correspondence to some kind of generalized algebras, like it was in $K3$ case. From the large number of possibly graphs we illustrated some selected cases corresponding generalized symmetric Berger matrices. It is very remarkable that some of these graphs can naturally be extended into infinite series while some others seem to remain “exceptional”.

What would be this new algebraic structure?, which would be the set of symmetries associated to this algebra?. It is very well known, by the Serre theorem, that Dynkin diagrams defines one-to-one Cartan matrices. In this work, we have generalized some of the properties of Cartan matrices for Cartan-Lie and Kac-Moody algebras into a new class of affine, and non-affine Berger matrices. We arrive then to the obvious conclusion that any algebraic structure emerging from this can not be a CLA or KMA algebra.

The main difference with previous definitions being in the values that diagonal elements of the matrices can take. In Calabi-Yau CY_3 spaces, new entries with norm equal to 3 are allowed. The choice of this number can be related to two facts: First, we should take in mind that in higher dimensional Calabi-Yau spaces resolution of singularities should be accomplished by more topologically complicated projective spaces: while for resolution of quotient singularities in $K3$ case one should use the CP^1 with Euler number 2, the, Euler number 3, CP^2 space could be use for resolution of some singularities in CY_3 space. The second fact is related to the cubic matrix theory[21], where a ternary operation is defined and in which the S_3 group naturally appears. One conjecture, draft from the fact of the underlying UCYA construction, is that, as Lie and affine Kac-Moody algebras are based on a binary composition law; the emerging picture from the consideration of these graphs could lead us to algebras including simultaneously different n-ary composition rules. Of course, the underlying UCYA construction could manifest in other ways: for example in giving a framework for a higher level linking of algebraic structures: Kac-Moody algebras among themselves and with any other hypothetical algebra generalizing them. Thus, putting together UCYA theory and graphs from reflexive polyhedra, we expect that iterative application of non-associative n-ary operations give us not only a complete picture of the RWV, but allow us in addition to establish “dynamical” links among RWV vector and graphs of different dimensions and, in a further step, links between singularity blow-up and possibly new generalized physical symmetries.

As it is mentioned in the last section, generalizations of the E_7^1 and E_8^1 series of graphs can also be easily obtained. It is suggestive to consider the procedure outlined in this section as a new way of unifying two semisimple algebras within a larger algebra. By the reverse process, the decomposition of the Berger diagram leads to proper, Kac-Moody, subdiagrams and the reduction of the Berger matrix to Kac-Moody blocks. This can be performed eliminating the nodes corresponding to one or more roots and the lines attached to them.

In particular, it is suggestive to consider the case where a Berger matrix resembling by blocks copies of $E_7^{(1)}, E_8^{(1)}$ matrices

$$G \sim CY3B(8_L + 2_{int} + 8_R) \leftarrow E_7^{(1)} + A_r + E_7^{(1)}$$

or

$$G \sim CY3B(9_L + 2_{int} + 9_R) \leftarrow E_8^{(1)} + A_r + E_8^{(1)}$$

are reduced to standard blocks. As a result of this reduction the single o affine condition or central vector is broken down to three different affine conditions for each of the blocks. We could compare this case with the heterotic string example where the unified gauge group is obtained by standard direct sum of algebras $G = E_8 + E_8$.

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